January 2010 Core 4 Written solutions

① $f(-1) = 15(-1)^3 + 19(-1)^2 - 4$

$= -15 + 19 - 4 = 0$ (∴ $(x+1)$ is a factor)

(ii) $f\left(\frac{2}{5}\right) = 15\left(\frac{2}{5}\right)^3 + 19\left(\frac{2}{5}\right)^2 - 4$

$= \frac{24}{25} + \frac{76}{25} - 4 = 0$

∴ $(5x-2)$ is a factor.

(b) $f(x) = (x+1)(5x-2)(ax+b) = (x+1)(5x-2)(3x+2)$

$\underline{a=3}$ $\underline{b=2}$

$$\frac{15x^2-6x}{f(x)} = \frac{3x\cancel{(5x-2)}}{(x+1)\cancel{(5x-2)}(3x+2)} = \frac{3x}{(x+1)(3x+2)}$$

② $\cos x + 3\sin x = R\cos(x-\alpha)$

$\cos(x-\alpha) = \cos x\cos\alpha + \sin x\sin\alpha$

$\cos x + 3\sin x = (R\cos\alpha)\cos x + (R\sin\alpha)\sin x$

$\Rightarrow 1 = R\cos\alpha$ ① $3 = R\sin\alpha$ ②

$\frac{②}{①}$: $3 = \frac{\sin\alpha}{\cos\alpha} \Rightarrow \tan\alpha = 3$

$\alpha = \tan^{-1}(3) = 1.249$

$①^2 + ②^2$: $1^2+3^2 = R^2\cos^2\alpha + R^2\sin^2\alpha$

$10 = R^2(\cos^2\alpha + \sin^2\alpha)$

$R^2 = 10 \Rightarrow R = \sqrt{10}$.

$\underline{\sqrt{10}\cos(x-1.249)}$

(b)(i) min value occurs at min value of $\cos(x - 1.249)$

so min value is $\underline{\underline{-\sqrt{10}}}$

(ii) $\cos\theta = -1$ when $\theta = \pi, -\pi$

$x - 1.249 = -\pi, \pi$

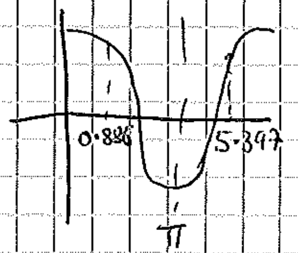
$x = \cancel{-1.893}, \ \underline{\underline{4.391}}$

(c) $\sqrt{10}\cos(x - 1.249) = 2$

$\cos(x - 1.249) = \dfrac{2}{\sqrt{10}}$

$x - 1.249 = 0.886, \ 5.397$

-0.886



$\underline{x = 0.363, \ 2.135}$

(others out of range).

③ (a) (i) $(1+x)^{-\frac{1}{3}} \approx 1 + \left(\frac{-1}{3}\right)x + \dfrac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)x^2}{2!}$

$= 1 - \dfrac{x}{3} + \dfrac{2}{9}x^2$

(ii) replace x with $\frac{3}{4}x$

$1 - \frac{1}{3}\left(\frac{3}{4}x\right) + \frac{2}{9}\left(\frac{3}{4}x\right)^2 = 1 - \frac{1}{4}x + \frac{1}{8}x^2$

(b) $\sqrt[3]{\dfrac{256}{4+3x}} = \sqrt{\dfrac{\frac{1}{4}(256)}{\frac{1}{4}(4+3x)}} = \sqrt[3]{\dfrac{64}{1+\frac{3}{4}x}} = 4\left(1+\frac{3}{4}x\right)^{-\frac{1}{3}}$

$\approx 4\left(1 - \frac{1}{4}x + \frac{1}{8}x^2\right) = 4 - x + \frac{1}{2}x^2$

$a = 4 \quad b = -1 \quad c = \frac{1}{2}$

4) $$\frac{10x^2+8}{(x+1)(5x-1)} = 2 + \frac{A}{(x+1)} + \frac{B}{(5x-1)}$$

$$= \frac{2(x+1)(5x-1)}{(x+1)(5x-1)} + \frac{A(5x-1)}{(x+1)(5x-1)} + \frac{B(x+1)}{(x+1)(5x-1)}$$

$$\Rightarrow \quad 10x^2+8 = 2(x+1)(5x-1) + A(5x-1) + B(x+1)$$

let $\underline{x=-1}$

$$10+8 = A(-6) \quad \Rightarrow \quad 18 = -6A \quad \Rightarrow \quad \underline{\underline{A=-3}}$$

let $\underline{x=\frac{1}{5}}$

$$10\left(\frac{1}{5}\right)^2 + 8 = B\left(\frac{6}{5}\right)$$

$$\frac{10}{25} + 8 = \frac{6}{5}B \qquad \frac{42}{\cancel{5}} = \frac{6}{\cancel{5}}B \qquad \Rightarrow \underline{\underline{B=7}}$$

$$2 + \frac{-3}{(x+1)} + \frac{7}{(5x-1)}$$

(b) $$\int 2 - \frac{3}{(x+1)} + \frac{7}{(5x-1)}\,dx = 2x - 3\ln(x+1) + \frac{7}{5}\ln(5x-1) + C$$

5) $$x^2 + xy = e^y$$

product rule

$$\frac{d}{dx}(xy) = \left(\frac{d}{dx}x\right) \times y + x\left(\frac{d}{dx}y\right) = y + x \times \frac{dy}{dx}\frac{d}{dy}y$$

$$= y + x\frac{dy}{dx}$$

$$\frac{d}{dx}\left(e^y\right) = \frac{dy}{dx}\frac{d}{dy}e^y = \frac{dy}{dx} \times e^y$$

$$\Rightarrow \frac{d}{dx}\left(x^2 + xy\right) = \frac{d}{dx}\left(e^y\right)$$

$$2x + y + x\frac{dy}{dx} = e^y\frac{dy}{dx}$$

$$2x + y = \frac{dy}{dx}\left(e^y - x\right)$$

$$\frac{dy}{dx} = \frac{2x+y}{e^y - x}$$

$$x = -1 \quad y = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{-2}{1+1} = \underline{\underline{-1}}$$

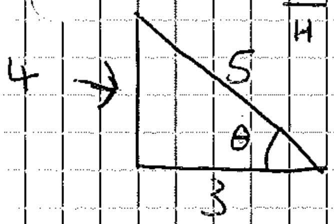
6 (i) $\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \sin\theta\cos\theta$

$$= 2\sin\theta\cos\theta$$

$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$$= \cos^2\theta - \sin^2\theta$$

(ii) $\cos\theta = \frac{3}{5} = \frac{A}{H} \Rightarrow \sin\theta = \frac{4}{5}$



$\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$

as required.

$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

$$= \frac{9}{25} - \frac{16}{25} = \frac{-7}{25}$$

(b) (i) $x = 3\sin 2\theta \qquad y = 4\cos 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\frac{dy}{d\theta} = -8\sin 2\theta \qquad \frac{dx}{d\theta} = 6\cos 2\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{6\cos 2\theta}$$

$$\frac{dy}{dx} = \frac{-8\sin 2\theta}{6\cos 2\theta} = \frac{-4}{3}\frac{\sin 2\theta}{\cos 2\theta}$$

$$= \frac{-4}{3}\tan(2\theta)$$

(ii) $\cos\theta = \frac{3}{5}$ from (a): $\sin 2\theta = \frac{24}{25} \qquad \cos 2\theta = \frac{-7}{25}$

$$\frac{dy}{dx} = \frac{-4}{3} \times \frac{24}{25} \times \frac{-25}{7} = \frac{96}{21} = \frac{32}{7}$$

$$x = 3\sin 2\theta = 3 \times \frac{24}{25} = \frac{72}{25}$$

$$y = 4\cos 2\theta = 4 \times \frac{-7}{25} = \frac{-28}{25}$$

$$y + \frac{28}{25} = \frac{32}{7}\left(x - \frac{72}{25}\right)$$

7. $\frac{dy}{dx} = \frac{1}{y}\cos\left(\frac{x}{3}\right)$ separate variables:

$$\int y\, dy = \int \cos\left(\frac{x}{3}\right) dx$$

$$\frac{1}{2}y^2 = 3\sin\left(\frac{x}{3}\right) + C$$

$$y^2 = 6\sin\left(\frac{x}{3}\right) + k$$

$y = 1 \qquad x = \frac{\pi}{2}$

$$1 = 6\sin\left(\frac{\pi}{6}\right) + k$$

$$\Rightarrow \underline{k = -2} \qquad y^2 = 6\sin\left(\frac{x}{3}\right) - 2$$

(8) $r = \begin{pmatrix} 2+\lambda \\ -1-3\lambda \\ -5+2\lambda \end{pmatrix}$

If B lies on r then $0 = 2+\lambda \Rightarrow \underline{\lambda = -2}$

check other coordinates:

$5 = -1-3\lambda$

$5 = -1-3(-2) = 5$ ✓

$-9 = -5+2(-2) = -5-4 = -9$ ✓

so B lies on L.

(b) $\vec{BC} = \vec{OC} - \vec{OB} = \begin{pmatrix} 9 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ -9 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ 12 \end{pmatrix}$

c)(i) $2\vec{BC} = \begin{pmatrix} 18 \\ -6 \\ 24 \end{pmatrix}$

$\vec{AD} = \vec{OD} - \vec{OA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} = \begin{pmatrix} x-2 \\ y+1 \\ z+5 \end{pmatrix}$

$x-2 = 18 \Rightarrow x = 20$

$y+1 = -6 \Rightarrow y = -7$ as required $D\begin{pmatrix} 20 \\ -7 \\ 19 \end{pmatrix}$

$z+5 = 24 \Rightarrow z = 19$

(ii) $P\begin{pmatrix} 2+p \\ -1-3p \\ -5+2p \end{pmatrix}$ $\vec{PD} = \vec{OD} - \vec{OP} = \begin{pmatrix} 20 \\ -7 \\ 19 \end{pmatrix} - \begin{pmatrix} 2+p \\ -1-3p \\ -5+2p \end{pmatrix}$

$\vec{PD} = \begin{pmatrix} 18-p \\ -6+3p \\ 24-2p \end{pmatrix}$

$\vec{PD} \perp L$ which means that

$$\begin{pmatrix} 18-p \\ -6+3p \\ 24-2p \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = 0$$

$$\Rightarrow \quad 18-p-3(-6+3p)+2(24-2p)=0$$

$$\begin{matrix} 18 & -p & \\ +18 & -9p & =0 \\ +48 & -4p & \end{matrix}$$

$$84 = 14p \quad \Rightarrow \quad \underline{\underline{p=6}}$$

(9)(i) $\underline{t=0}$ $\quad h = A(1-e^0) = 0.$

(ii) $t=12 \quad h=57$

$$57 = A(1-e^{-3}) \quad \Rightarrow \quad A = \frac{57}{1-e^{-3}} = 59.987$$

$$= 60 \ (2sf).$$

(b)(i) $h=48.$

$$48 = 60(1-e^{-\frac{1}{4}t}) \quad \Rightarrow \quad 1-e^{-\frac{1}{4}t} = \frac{48}{60}$$

$$e^{-\frac{1}{4}t} = 1-\frac{4}{5} = \frac{1}{5}$$

$$\ln(e^{-\frac{1}{4}t}) = \ln\left(\frac{1}{5}\right)$$

$$-\frac{1}{4}t = \ln\left(\frac{1}{5}\right)$$

$$t = -4\ln\left(\frac{1}{5}\right) = 4\ln\left(\frac{1}{5}\right)^{-1} = \underline{\underline{4\ln 5}}$$

(ii) $h = 60 - 60e^{-\frac{1}{4}t}$

$$\frac{dh}{dt} = -60 \times \left(\frac{-1}{4}\right) e^{-\frac{1}{4}t}$$

$$= 15e^{-\frac{1}{4}t}$$

$$\frac{h}{4} = 15 - 15e^{-\frac{1}{4}t}$$

$$\frac{h}{4} = 15 - \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = 15 - \frac{h}{4}$$

(iii) at $\frac{dh}{dt} = 13$

$$13 = 15 - \frac{h}{4}$$

$$\frac{h}{4} = 2 \quad \Rightarrow \quad \underline{\underline{h = 8}}$$